

# Alice and Bob and Hendrik

Rafael Bautista-Mena

CEIBA Center, Universidad de los Andes, Bogotá, Colombia

February 5, 2008

## Abstract

This paper offers an alternative approach to discussing both the principle of relativity and the derivation of the Lorentz transformations. This approach uses the idea that there may not be a preferred inertial frame through a privileged access to information about events. In classroom discussions, it has been my experience that this approach produces some lively arguments.

## 1 Concepts and conventions

Suppose that two inertial “observers”, from now on named Alice and Bob, each attached to a reference frame in one spatial dimension and time, need to exchange information about an event  $E$  that is a part of an information set to which both have access to. In discussions about kinematics, the information set about  $E$  reduces to three items: 1.- The fact that it took place (for instance, as registered by a “click” in standard detectors that both Alice and Bob are endowed with.) 2.- The spatial coordinate for  $E$ , and 3.- the associated time. In such discussions, the first item is almost always taken for granted. For the purposes of this work, the only proviso I will make is to call “events” only those described in item 1, and avoid the more widespread usage, where “event” tends to be more or less automatically identified with a space-time point. Items 2 and 3 are the usual concern of kinematics, and as is well known, the communication of these data between Alice and Bob goes via a set of simple, linear transformation equations, namely the Lorentz transformations.

Another element necessary for the discussion that follows is to assume the existence of some standard “messenger”, which may be produced by any event, and is by definition the fastest<sup>1</sup> entity known to the observers to be apt to carry encoded messages across empty space. As an integral part of the definition of their frames, Alice and Bob are capable of either encoding or decoding messages, using some universal code. Such messages always carry the information about those events registered in their frames. By assumption, any event that Alice is capable of registering with her detector will also be detectable by Bob.

Bob’s frame of reference will be drawn as a Cartesian frame, with the vertical time axis perpendicular to the horizontal space axis. By definition, the origin corresponds to the event of coincidence with that of Alice’s frame. This event is labelled  $O$  in Figure 1. Alice’s choice of axes for encoding her information about events is completely defined by the angles (see Figure 1)  $\alpha$  and  $\gamma$  that her time and space axes respectively make with those of Bob’s frame. I adopt the convention that, as drawn, these angles are positive, and satisfy the constraint

$$\alpha + \gamma \leq \frac{\pi}{2} \quad (1)$$

Aside from this constraint, Alice’s choice of frame would in principle be arbitrary.

Measured from Alice’s time axis, the world lines followed by the messenger in her frame tilt an angle  $\beta$ . All lines parallel to Alice’s  $x$  axis will be called “equitemps”, and all lines parallel to her time axis will

---

<sup>1</sup>As measured in each of their frames.

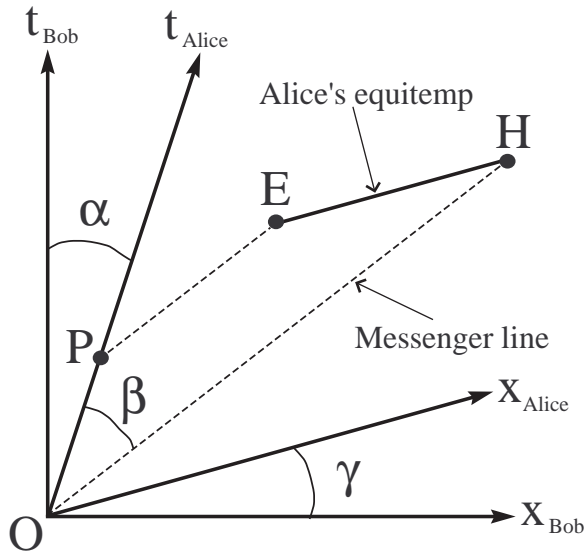


Figure 1: Angle conventions between Alice's and Bob's frames

be called “equilocs”, after the denomination used in Mermin [1].

It is important to emphasize that the particular choice of perpendicular axes for Bob's frame is made only for ease of exposition. In fact, all that matters is the relative angle between Bob's and Alice's corresponding axes. In what follows, methods of Euclidean geometry will be used within a context that is usually associated with Minkowski's space-time. Further details as to why and how this may be done can be found in Brill and Jacobson [2].

### 1.1 Locality of the observers

The act of interpreting, or decoding, a message is, by assumption, local in character. I will model this assumption by locating Alice at some definite, fixed point. By convention, Alice will be located at the spatial origin for all time. As a consequence, Alice will learn about the occurrence of event  $E$ , that took place say at frame coordinates  $(x_E, t_E)$  only at a later

time  $t_E + x_E/c$ , where  $c$  is the speed of the standard messenger in her frame. In a symmetric fashion, if Alice wanted to be causally connected with event  $E$ , then the latest moment at which she could send a “triggering” message would be  $t_E - x_E/c$ . This particular event, Alice's delivery of the latest signal that could connect her to  $E$ , is shown at the vertex  $P$  in Figure 1. As an obvious extension, Alice could be causally connected to any event that may be triggered along the world-line connecting Alice with event  $E$ . Another event of interest is the “horizon” event  $H$ . This is the event simultaneous with  $E$ , triggered by a messenger sent out from  $O$  along the positive  $x$  axis. Notice that this is the farthest point to which Alice may expect to be causally connected with before or simultaneously with event  $E$ .

### 1.2 Accessible sets

An accessible event, conditional on events  $E$  and  $O$ , is one for which Alice may be able to have a causal connection with before or at most simultaneously with the occurrence of event  $E$ . The accessible set  $A$ , conditional on events  $O$  and  $E$ , is the set of all space-time points with which Alice could establish a causal connection, right after her time  $t = 0$  and up until the frame time at which  $E$  takes place. From this description, it is clear that  $E$  must lie within the “causal cone” defined by the messenger. In the space-time diagram shown in Figure 1, the polygon  $OPEHO$  corresponds to the accessible set conditional on events  $O$  and  $E$ .

The intuition for  $A$  is to think of it as composed by member “sites”, all identical in their properties, and distinguished only by their space-time coordinates. Each member of  $A$  is equally capable of hosting a single event.<sup>2</sup> Therefore,  $A$  describes Alice's capacity to influence events along the positive  $x$  axis between  $t = 0$  and the time corresponding to event  $E$ . Alter-

<sup>2</sup>Here the “site” of an event is assumed to correspond to a single member of the set. In principle, one could consider the possibility of a larger subset of  $A$  as the site of a single event, in which case the individual space-time coordinates of the elements in the subset would fail to provide any meaningful information about the event. But this case looks more like quantum physics. The present discussion is fully contained within a classical context.

natively,  $A$  may also be seen as the set containing the maximum amount of information (potential events) generated between those times, that Alice may expect to collect.

## 2 The relativity principle

Given the same constraints: a common event  $O$ , and an external, independent event  $E$ , no inertial observer may expect to be causally connected to more sites, or to be able to have access to more information than any other. In more mundane terms, given the same prior information, Alice may not know anything that Bob wouldn't know too, nor vice versa.

One way to make operative this form of the relativity principle is to assign a measure  $I(A)$  to set  $A$ . I will make the following assumption:

*The Euclidean area of the accessible set bounded by the polygon  $OPEHO$  is a direct measure of the maximum number of events about which Alice may have knowledge, conditional on events  $O$  and  $E$ .<sup>3</sup>*

Without providing a proof, it seems reasonable to suppose that this statement is fully consistent with the properties of homogeneity and isotropy of flat space-time.

## 3 Derivation of results

Let's begin by computing the coordinates for all four events defining Alice's accessible set, as determined by Bob. Figure 1 shows the polygon  $OPEHO$  and the associated angles.

In self-evident notation, the coordinates for each vertex as functions of Bob's coordinates  $(X, T)$  for event

$E$  are given by:

$$x_P = \frac{\sin \alpha}{\sin \beta} [T \sin(\alpha + \beta) - X \cos(\alpha + \beta)], \quad (2)$$

$$t_P = x_P \cot \alpha, \quad (3)$$

$$x_H = \frac{T - X \tan \gamma}{\cot(\alpha + \beta) - \tan \gamma}, \quad (4)$$

$$t_H = x_H \cot(\alpha + \beta). \quad (5)$$

The measure of the accessible set, corresponding to the area bounded by  $OPEHO$  is

$$I(A) = \frac{1}{2} [X t_P - T x_P + x_H T - t_H X]. \quad (6)$$

Equation (6) may be rewritten as follows:

$$2I(A) = h_1 T^2 + h_2 X^2 + h_3 X T. \quad (7)$$

The principle of relativity, as stated here, now requires that this measure be frame invariant. In other words, the  $h_i$ 's in (7) ought to be universal constants. It is not meant here that those coefficients are new physical constants, in the sense that Planck's constant or the charge of the electron are. But rather, that the corresponding algebraic expressions for the  $h_i$ 's must reduce, in a trivial way, to simple numerical values. Therefore, as an immediate consequence of the relativity principle, there follows:

$$h_i = \text{constant}. \quad (8)$$

Their explicit forms are the following:

$$h_1 = -\frac{\sin \alpha}{\sin \beta} \sin(\alpha + \beta) + \frac{1}{\cot(\alpha + \beta) - \tan \gamma}, \quad (9)$$

$$h_2 = -\frac{\cos \alpha}{\sin \beta} \cos(\alpha + \beta) + \frac{\cot(\alpha + \beta) \tan \gamma}{\cot(\alpha + \beta) - \tan \gamma}, \quad (10)$$

<sup>3</sup>I am aware that this implies attaching to the set  $A$ , and to space-time in general, a topology different from the usually assumed for continuous space-time. In fact, it would have to be based on finite, or at most, countable sets. But the size of the corresponding "space-time cells" could be made as small as desired, as long as they were finite.

$$h_3 = \frac{\cos \alpha \sin(\alpha + \beta) + \sin \alpha \cos(\alpha + \beta)}{\sin \beta} - \frac{\cot(\alpha + \beta) + \tan \gamma}{\cot(\alpha + \beta) - \tan \gamma}. \quad (11)$$

These rather lengthy expressions may be more easily handled using the following shorthand notation:  $v \equiv \tan \alpha$ ;  $w \equiv \tan(\alpha + \beta)$ ;  $z \equiv \tan \gamma$ . Now (9), (10) and (11) look as follows:

$$h_1 = -\frac{wv}{w-v} + \frac{w}{1-zw}, \quad (12)$$

$$h_2 = -\frac{1}{w-v} + \frac{z}{1-zw}, \quad (13)$$

$$h_3 = \frac{w+v}{w-v} - \frac{1+zw}{1-zw}. \quad (14)$$

Inspection of (12), (13) and (14) leads to the following identity:

$$\frac{h_1}{w} + h_2w + h_3 = 0. \quad (15)$$

From the statement of the relativity principle in (8), equation (15) implies that  $w$  is equal to some constant value. Therefore:

$$\alpha + \beta = \text{constant}. \quad (16)$$

Then, irrespective of the choice of frame, the relative slope associated with the speed of the messenger is fixed.

Equation (12) can be rearranged as follows:

$$vz(w^2 - h_1w) + w^2h_1z + (h_1 - 2w)v + w^2 - h_1w = 0. \quad (17)$$

Since both  $v$  and  $z$  represent trigonometric functions, and since (17) must be an identity, quadratic terms should be linearly independent from linear terms, therefore the coefficient in the quadratic term must vanish, leaving as its only feasible solution:

$$w = h_1. \quad (18)$$

Notice that  $w = 0$  is not a feasible solution, for it doesn't solve (15). With this result, equation (17) reduces to:

$$h_1^2z - v = 0. \quad (19)$$

From this relation follows that, if the principle as stated by (8) is to be upheld, then the choice of axes by Alice is constrained by (19). This relationship is, by the way, the best justification of why  $h_1$  must be different from zero, for otherwise, Alice wouldn't have a choice at all, or put another way, it would deny the existence of any reference frame.

Using (18), equation (15) becomes:

$$1 + h_1h_2 + h_3 = 0. \quad (20)$$

Substitution of (18) and (19) into (13) yields:

$$h_1h_2 = -1. \quad (21)$$

This last result, combined with (20) produce:

$$h_3 = 0. \quad (22)$$

These findings for the  $h_i$ 's lead back to (7), which now reduces to:

$$2I(A) = h_1X^2 - \frac{T^2}{h_1}. \quad (23)$$

In this expression, it is always possible to set  $h_1 = 1$ , because this is just a rescaling of the ruler and the "tick" of the clock used by Alice. Then (23) is easily recognizable as the Minkowski square of the space-time interval. This same choice makes  $w = 1$ , which, going back to (16), produces the neat result:

$$\alpha + \beta = \frac{\pi}{4}. \quad (24)$$

Then, the messenger's slope must cut in halves the quadrant of Bob's frame. Finally, (19) simplifies to:

$$v = z. \quad (25)$$

Using the convention established earlier for  $\alpha$  and  $\gamma$ , the last equation is equivalent to say that they are equal. Therefore, Alice's axes are also placed symmetrically around the line of the messenger. This geometrical arrangement is well known: Bob's and Alice's frames are connected by the Lorentz transformation.

The second consequence that follows from (25) is that it makes obvious that in Alice's frame the speed of the messenger is the same as in Bob's frame. Therefore, there exists one messenger whose speed is the same in all frames of reference.

## 4 Discussion

In the present work I have derived both the necessity of the existence of a messenger with an invariant speed in all frames of reference and the Lorentz transformations, starting from the principle of relativity, stated as a symmetry in the access to causal connections. In more relaxed terms, this approach establishes the impossibility to tell the state of inertial motion via the access to different “amounts of information” between reference frames. This approach relies on two assumptions:

1. The local character of any encoding/decoding capable “observer”.
2. The measure  $I(A)$  as the correct invariant quantity.

Traditionally, the question ‘Why the Lorentz transformation?’ has been answered with ‘Because it is the only solution consistent with the relativity principle.’ The present work instead addresses the question ‘Which way to the relativity principle?’ Within the context of this paper, the principle has been spelled out through the invariance of the measure of the conditional accessible set  $I(A)$ .

A question raised by this approach may be why it works. It is not new to obtain the Lorentz transformations from the relativity principle, plus additional assumptions about the properties of flat space-time, as it has been shown in several excellent articles (see, for instance, Lévy-Leblond [4], Mermin [3], Lee and Kalotas [5].) The only difference in my approach is the expression of the principle in terms of a kind of information democracy, which is closer in spirit to the intend in Field [7], who arrives at the Lorentz transformations from a postulated space-time exchange invariance. To see the connection with other treatments, recall that a universal messenger generates a causal ordering on the future cone. Therefore, the relativity principle imposes a causal structure on set  $A$ . Turning this argument around, suppose now that we would want to have a set  $A$  with a postulated causal structure.<sup>4</sup> Suppose

---

<sup>4</sup>See an interesting approach to the Lorentz transformations from this angle in [6].

also that Alice triggered an event timed between events  $O$  and  $E$ . Since Alice and Bob are equivalent in their capacity to register events, Bob would learn about such event. But he could not register this event as having occurred either before  $O$  or after  $E$ , because that would violate the assumption of a causal structure for  $A$ . Therefore, all the events that Alice would trigger between  $O$  and  $E$ , are the same ones that Bob could detect too, no more and no less. This argument sheds light on why the invariance of the measure  $I(A)$  acts as a substitute for the conventional statement of the relativity principle. Then, the main contribution of this particular formulation is its approach to relativity from an event-counting concept, represented (as proxy) by a Euclidean measure. On the other hand, one limitation is that it starts from the assumption that the correct transformation relation between inertial frames is linear.

This approach is open to criticism, among other reasons, on the basis that it looks like a step back toward anthropocentrism, through my recourse to terms such as “information”, “encoding”, “decoding”, and others of a similar nature. I always bear in mind the now famous retort ‘Whose information?’ Nevertheless, I believe that my use of such terms only highlights the limitations of language. After all, in the case of, say, an elastic collision between two electrons, we use terms such as “interaction” to refer to the exchange of momentum between the particles, only out of well established tradition.

## 5 Acknowledgements

I wish to thank Ana Rey, Juan Restrepo, Alonso Botero and Jorge Villalobos for their useful comments and their kind help.

## References

- [1] N.D. Mermin, “From Einstein’s 1905 Postulates to the Geometry of Flat Space-Time”, gr-qc/0411069.

- [2] D. Brill and T. Jacobson, “Spacetime and Euclidean Geometry”, gr-qc/0407022.
- [3] N.D. Mermin, “Relativity without light”, American Journal of Physics 52, 119-124 (1984).
- [4] J-M. Lévy-Leblond, “One more derivation of the Lorentz transformations”, American Journal of Physics 44, 271-277 (1976).
- [5] A.R. Lee and T. M. Kalotas, “Lorentz transformation from the first postulate”, American Journal of Physics 43, 434-437 (1975).
- [6] E.C. Zeeman, “Causality implies the Lorentz group”, J. Math. Phys. 5, 490-493 (1964).
- [7] J.H. Field, “Space-time exchange invariance: Special relativity as a symmetry principle”, American Journal of Physics 69, 569-575 (2001).